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SHORTER COMMUNICATION

HEATING OF A CYLINDRICAL CAVITY

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NOMENCLATURE

a, radius of cavity;
C, $\exp(\gamma)$;
d_m, factors in small τ expansion of $\bar{\theta}_s$;
$E^{(p)}$, multiple Euler transformation of a series;
$e_m^{(p)}$, terms of $E^{(p)}$ series;
F_∞, asymptotic expansion of $\bar{\theta}_s(\tau)$;
k, thermal conductivity;
$\bar{Q}$, a characteristic heat-transfer rate;
Q, $\bar{Q}\phi(\tau)$;
r, radius;
T_0, initial temperature;
T, temperature;
t, time.

Greek symbols

α, thermal diffusivity;
Γ, gamma function;
γ, Euler's constant;
θ, $2(T-T_0)k/(\bar{Q}a)$;
$\bar{\theta}$, value of θ for constant heating problem;
θ_s, $\theta(r=a)$;
$\bar{\theta}_s$, $\bar{\theta}(r=a)$;
λ_m, coefficients in small τ expansion of $\bar{\theta}_s$;
ϕ, a dimensionless surface heat-transfer rate;
ϕ_m, coefficient in small τ expansion of θ_s, see equation (5);
τ, $\alpha t/a^2$;
ξ, dummy variable.

1. INTRODUCTION

THE SOLUTION to the problem of uniform heating, $\bar{Q}$, of a cylindrical cavity wall of radius $r = a$ where the material space $r > a$ is initially (time $t = 0$) at the uniform temperature T_0, is presented in [1]. The solution for the dimensionless temperature $\bar{\theta} = 2(T-T_0)k/(\bar{Q}a)$ (where k is the thermal conductivity) is given there in integral form. Leading terms in expansions for small and large dimensionless time $\tau = \alpha t/a^2$ (where α is the thermal diffusivity) are also obtained in [1]. These latter expansions do not share a common range of convergence. Accordingly, for moderate τ, some results for $\bar{\theta}$ have been computed from numerical integration of the integral solution. The results are given in tabular form [2] and, for the cavity surface, they are plotted in [1].

In the usual manner, the utility of a step disturbance solution becomes significantly extended by taking advantage of the linearity of the problem, e.g. through the use of Duhamel's integral to solve for transient problems with time varying surface heating. To follow such an attack in the present problem, i.e. to solve the problem for specific time varying surface cavity heating, one is required in one way or another to utilize the above mentioned tabular type presentation of the solution to the fundamental, constant surface heating problem. The whole procedure would be more attractive if an analytic representation for the fundamental solution valid for all $\tau > 0$ could be obtained. In this regard one might hope, for example, to extend the useful range of the τ expansion of $\bar{\theta}$ into moderate to large τ values, i.e. into the τ range where the asymptotic, large τ expansion of $\bar{\theta}$ is useful. In the present problem this cannot be done simply by obtaining a large number of terms in the small τ expansion. Indeed, even if this latter expansion is convergent in some neighborhood of $\tau = 0$ it appears that the radius of convergence is severely restricted to a range $|\tau| \ll 1$. Nevertheless, by using an appropriate transformation of the terms of the small τ expansion, useful results far beyond the above mentioned small τ range can be extracted.

The situation has been studied for the particularly interesting temperature history of the cavity surface. For this surface, application of the Euler transformation [3, 4] to the original small τ series representation of $\bar{\theta}$ has proven to be useful. Details of this application are presented in the next section. Finally, in the last section, the Euler transformation idea is extended to the case of time dependent surface heating and results are obtained in the example problem $Q(\tau) \sim \tau^q$.

2. CAVITY SURFACE TEMPERATURE—UNIFORM HEATING

Following [1] and extending the results presented there, the following expansions of $\bar{\theta}_s(\tau) = \bar{\theta}(r/a = 1, \tau)$ for large and small τ, have been obtained

$$\lim_{\tau\to\infty} \bar{\theta}_s = F_\infty(\tau) = \ln(4\tau/C) + \ln(4\tau/C)/(2\tau) + 1/(2\tau) - 3\ln^2(4\tau/C)/(16\tau^2) - \ln(4\tau/C)/(16\tau^2) + (\pi^2+3)/(32\tau^2) + O(\ln^3\tau/\tau^3) \quad (1)$$

$$\bar{\theta}_s = \sum_{m=1}^{\infty} d_m \tau^{m/2}/\Gamma(1+m/2)$$

$$= \sum_{m=1}^{\infty} \lambda_m \tau^{m/2} = 4\tau^{\frac{1}{2}}/\pi^{\frac{1}{2}} - \tau + \ldots \quad (2)$$

where $d_1 = 2$, $d_{m+1} = \dfrac{(-1)^m[(2m-1)!]^2}{2^{5m-2}m[(m-1)!]^3} + \sum_{n=1}^{m} \dfrac{(-1)^n(2n+1)!(2n-2)!}{2^{5n-1}[(n-1)!]^3 n^2} d_{m-n+1}\ m > 0.$

and where $\Gamma(x)$ is the Gamma function and $\ln C = \gamma = 0.5772\ldots$ is Euler's constant.

.Following the above ideas of Section 1, the Euler transformation is introduced and applied to the expansion of equation (2). Thus, instead of approximating the infinite sum of equation (2) by generally useless (except for very small τ) but rationally obtained partial sums we will approximate $\bar{\theta}_s$ by

$$\bar{\theta}_s \simeq E^{(p)}\left[\sum_{m=1}^{M} \lambda_m \tau^{m/2}\right] = \sum_{m=1}^{M} e_m^{(p)}(\tau) \quad p > 0, \quad (3)$$

where

$$e_m^{(0)} = \lambda_m \tau^{m/2},\ e_m^{(p)} = \frac{1}{2^m}\sum_{n=1}^{m} \frac{(m-1)!}{(m-n)!(n-1)!} e_{m-n+1}^{(p-1)}\ p > 0.$$

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Table 1. Computation from equation (3) for surface temperature $\bar{\theta}_s$ due to constant surface heating and for $\tau = 10$ and $\tau = 30$

$\tau = 30$

M	$E^{(1)}[S_M]$	Error estimate %	$E^{(2)}[S_M]$	Error estimate %	$E^{(3)}[S_M]$	Error estimate %	$E^{(4)}[S_M]$	Error estimate %
1	6.18	—	3.09	—	1.55	—	$7.73(10^{-1})$	—
2	1.77	$> O(100)$	3.53	$O(12)$	2.43	†	1.38	†
3	7.40	*	3.90	$O(10)$	2.97	†	1.86	†
4	−1.16		3.99	$O(6)$	3.32	†	2.25	†
5	$1.63(10^1)$		4.16	*	3.56	†	2.56	†
6	$2.87(10^1)$		4.12		3.73	$O(6)$	2.81	†
7	$1.17(10^2)$		4.31		3.85	$O(4)$	3.02	†
8	$-3.96(10^2)$		4.06		3.94	$O(3)$	3.20	†
9	$1.67(10^3)$		4.62		4.01	$O(2)$	3.34	†
10	$-7.59(10^3)$		3.48		4.06	$O(1.5)$	3.46	†

Exact $\bar{\theta}_s(30) = 4.29$

Estimated $\bar{\theta}_s(30) = 4.06$ $\begin{cases}\text{Estimated error} = O(2)\% \\ \text{Exact error} = 5.5\%\end{cases}$

$\tau = 10$

M	$E^{(1)}[S_M]$	Error estimate %	$E^{(2)}[S_M]$	Error estimate %	$E^{(3)}[S_M]$	Error estimate %
1	3.57	—	1.78	—	0.45	—
2	2.85	$O(75)$	2.50	†	0.82	†
3	3.47	$O(19)$	2.84	†	1.15	†
4	3.05	$O(17)$	3.02	$O(7)$	1.42	†
5	3.64	*	3.12	$O(4)$	1.66	†
6	2.79		3.18	$O(3)$	1.86	†
7	4.28		3.22	$O(2)$	2.04	†
8	1.19		3.25	$O(1)$	2.19	†
9	8.34		3.27	$O(0.6)$	2.32	†
10	−9.82		3.28	$O(0.4)$	2.44	†

Exact $\bar{\theta}_s(10) = 3.30$

Estimated $\bar{\theta}_s(10) = 3.28$ $\begin{cases}\text{Estimated error} = O(0.4)\% \\ \text{Exact error} = 0.87\%\end{cases}$

*Series is asymptotic and it was truncated after previous term.
†Estimate is less accurate than previous best estimate.

Here $E^{(p)}$ is the p'th Euler transformation of partial sum of the series of equation (2). The whole idea of the Euler transformation is to render convergence, more rapid convergence, or more rapid asymptotic convergence to a divergent, slowly convergent, or asymptotically convergent expansion. A theoretical basis for the success of the transformation is discussed in [3]. If, for a particular p, the series of equation (3) will yield a useful approximation to $\bar{\theta}_s$ then $e_m^{(p)}(\tau)$ will generally decrease with increasing m. This decrease will be monotonic if the series is convergent or monotonic up to a particular m if the series is asymptotic (at which m the series is truncated). Further, an error estimate can be based on the order of magnitude of the "latter terms", of this series. (In our application the order of magnitude of this error was taken to be the average of the orders of magnitude of the last two terms in the partial sum. Also, for a given approximation and error estimate pair to be valid it was required that it be consistent with the best approximation obtained with equation (3) and with the $p-1$ transformation.) Fixing τ and the number of terms, M, in the partial sum of equation (3), there is an optimum number of transformations, p, above which the estimate of equation (3) starts to deteriorate.

$\bar{\theta}_s$ was computed from equation (3) for different values of τ. The results of some of these calculations with $M = 1$ to $M = 10$ are presented in Table 1. In accordance with the above remarks, and for every τ considered, the suggested best estimate for $\bar{\theta}_s$ and its estimated error is finally given there along with the exact error (deduced from the aforementioned results given in [2]).

The most dramatic result that can be noted in Table 1 is that the leading ten terms of the τ expansion of equation (2) (which heretofore were totally useless in approximating $\bar{\theta}_s$ even to $\tau = O(1)$) have, under the multiple Euler transformation, yielded an estimate for $\bar{\theta}_s(\tau = 30)$ accurate to within 6% of the exact value. Furthermore, for each τ considered in the table the error estimate procedure suggested above has consistently yielded results in accordance with available exact results. It is of interest to note that increasing M to 19 and p to 3 the estimated error at $\tau = 30$ is minimized and equation (3) yields a result for τ accurate to 1.0%. Further increase in M and p do not reduce the estimated error.

Relative to accurate estimates for $\bar{\theta}_s(\tau)$ for all $\tau > 0$, perhaps the most useful result indicated in Table 1 is that in the range $\tau < 10$, equation (3) (with $p = 2$ and $M = 10$) leads to $\bar{\theta}_s$ estimates accurate to within 0.9% of the exact values. This result is to be complemented with the result that the large τ asymptotic representation for $\bar{\theta}_s$, equation (1), yields estimates to within O (0.5%) of exact values in the interval $\tau \geqslant 10$.

3. CAVITY SURFACE TEMPERATURE FOR VARIABLE HEATING—AN EXAMPLE

Consider the problem of heating the initially uniform temperature solid $r > a$ with a variable cavity surface flux

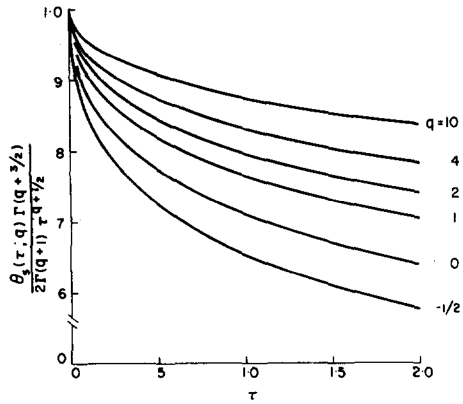

FIG. 1. Plot of $\theta_s(\tau; q)$ in the range $0 < \tau < 2$ for different values of q.

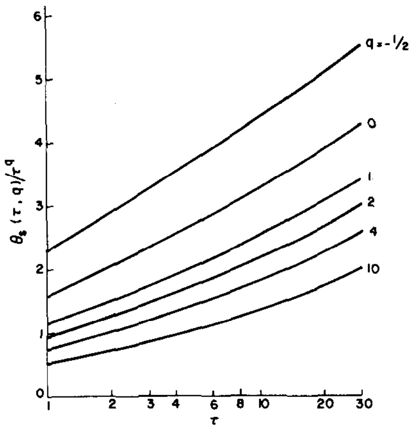

FIG. 2. Plot of $\theta_s(\tau; q)$ in the range $1 < \tau < 30$ for different values of q.

$Q(\tau) = \bar{Q}\phi(\tau)$, where Q is some characteristic heat flux. Then, if

$$\lim_{\tau \to 0} \phi(\tau) = O(\tau^q), \text{ where } q \geqslant -\tfrac{1}{2}, \text{ and if } \int_0^\tau \phi(\xi)\,d\xi$$

is bounded, the use of Duhamel's integral results in the following equation for the cavity surface temperature history

$$\theta_s(\tau) = \int_0^\tau \phi(\tau - \xi) \frac{d}{d\xi} \bar{\theta}_s(\xi)\,d\xi. \tag{4}$$

Here $\bar{\theta}_s(\tau)$ represents the fundamental constant surface heating solution of the previous section. Accordingly, for moderate τ, using a representation for $\bar{\theta}_s$ per equation (3) in the above equation (4) is suggested. Doing this yields the following general solution representation for $\theta_s(\tau)$

$$\theta_s(\tau) = E^{(p)}\left[\sum_{m=1}^{M} (m/2)\lambda_m \phi_m(\tau)\right] \tag{5}$$

where

$$\phi_m(\tau) = \int_0^\tau \phi(\tau - \xi)\xi^{m/2-1}\,d\xi.$$

It is expected that this latter result would yield a useful estimate for $\theta_s(\tau)$ in the small to moderate τ range.

As an example consider the class of heating $\phi(\tau) = \tau^q$, $q \geqslant -\frac{1}{2}$. Then from equation (5) we obtain

$$\theta_s(\tau; q) = [\tau^q \Gamma(q+1)/2] E^{(p)}\left[\sum_{m=1}^{M} \frac{\Gamma(m/2) m \lambda_m}{\Gamma(q+1+m/2)} \tau^{m/2}\right] \tag{6}$$

which we expect to yield a useful approximation for moderate to small τ. Besides the above result, equations (1) and (2) together with equation (4) can be used to obtain the following large τ results

$$\lim_{\tau \to \infty} \theta_s(\tau; q) = \tau^q \ln \tau [1 + O(1)]. \tag{7}$$

$\theta_s(\tau; q)$ has been computed from equation (6) for different q's, $q \geqslant -\frac{1}{2}$. In these calculations M and p were taken to be $M < 25$, $p < 4$. Plots of θ_s are presented in Fig. 1 for $0 < \tau < 2$. Plots of θ_s are presented in Fig. 2 for $1 < \tau < 30$.

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